A 2D BEM-FEM model of thin structures for time harmonic fluid-soil-structure interaction analysis including poroelastic media

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Abstract. This paper presents a two-dimensional BEM-FEM model of thin structures for time harmonic analysis when they are surrounded by inviscid fluid, viscoelastic and/or poroelastic media. The thin structures are considered as beams under the Euler-Bernoulli hypotheses, which are discretized by the FEM. The surrounding media are discretized by the BEM, where thin structures are seen as null thickness inclusions. The usage of the conventional SBIE for null thickness inclusions leads to a singular system of equations. To overcome this difficulty, the SBIE/HBIE dual formulation is used since it is the most direct approach. The HBIE and the SBIE/HBIE dual formulation for inviscid fluids and viscoelastic solids are well known, but not in the case of poroelastic solids. For this type of medium, a regularized form of the HBIE has been derived, which together with the SBIE/HBIE dual formulation are briefly presented. Also, appropriate equilibrium and compatibility conditions that couple the BEM equations (surrounding media) and the FEM equations (thin structures) are shown. This BEM-FEM model is validated against BEM-BEM models by studying a water reservoir where bottom sediments support a wall.

Introduction

This work presents a two-dimensional BEM-FEM dynamic model of thin structures surrounded by inviscid fluid, viscoelastic and/or poroelastic media. Thin structures buried or immersed in these types of media can be found in many applications: noise barriers, tunnels, retaining walls, sheet piles, slurry walls, etc.. The dynamic analysis of such structures can be performed using pure BEM or FEM models. However, these pure models face many difficulties. The BEM demands a mesh carefully executed, where the size of elements greatly depends on the thickness of the structure and the quasi-singular integration capabilities of the implementation. The FEM requires an important volume mesh for the surrounding media, and besides, it is unable to naturally incorporate the Sommerfeld radiation condition. This BEM-FEM model properly combines both methods and reduces these difficulties. It has been presented already for inviscid fluids [1].

The key of this approach is assuming that the thin structures are seen by the surrounding media as null thickness inclusions, being the surrounded media treated by the BEM and the thin structures by the structural FEM. It is well known that null thickness inclusions leads to a BEM degenerate system of equations if only SBIE (Singular Boundary Integral Equations) are used [2]. This can be avoided by the multiregion approach [3], which requires fictitious boundaries that increase the number of degrees of freedom. The most direct approach is the SBIE/HBIE (Hypersingular BIE) dual boundary formulation [4], whose difficulty lies on expressing the HBIE in an evaluable form. The HBIE can be treated by many ways [5], of which the reduction of the HBIE to a set of regular or weakly singular integrals is used in this work. The particular HBIE formulation has been already developed for elastostatics [6] and elastodynamics [7], and for inviscid fluids [1]. In this work, we briefly present the HBIE and the SBIE/HBIE dual formulation for the Biot's poroelastic media. The thin structures are considered as beams under the Euler-Bernoulli hypotheses with added rotational inertia. The FEM matrices have already been presented [8,1].

SBIE and HBIE for poroelastic media

The poroelastic media is assumed to be a fluid-filled poroelastic material governed by Biot's equations. The following formulation uses the notation and procedures presented in [9], but for the two-dimensional problem.

Let Ω be a poroelastic region, and Γ its boundary ($\Gamma=\partial\Omega$) with an orientation defined by its outward unit normal $\mathbf{n}$. The SBIE for a collocation point $\mathbf{x}_i \in \Omega$ can be written as:

$$\begin{bmatrix} J & 0 \\ 0 & \delta_{lk} \end{bmatrix}\begin{Bmatrix} \tau^i \\ u_k^i \end{Bmatrix} + \int_\Gamma \begin{bmatrix} -\left(U^*_{n00} + JX'^*_j n_j\right) & t^*_{0k} \\ -U^*_{nl0} & t^*_{lk} \end{bmatrix}\begin{Bmatrix} \tau \\ u_k \end{Bmatrix} \mathrm{d}\Gamma = \int_\Gamma \begin{bmatrix} -\tau^*_{00} & u^*_{0k} \\ -\tau^*_{l0} & u^*_{lk} \end{bmatrix}\begin{Bmatrix} U_n \\ t_k \end{Bmatrix} \mathrm{d}\Gamma, \quad \mathbf{C}^i\mathbf{u}^i + \int_\Gamma \mathbf{T}^*\mathbf{u}\,\mathrm{d}\Gamma = \int_\Gamma \mathbf{U}^*\mathbf{t}\,\mathrm{d}\Gamma \tag{1}$$

The fluid equivalent stress τ and the displacements u_k of the solid skeleton are gathered in the vector $\mathbf{u}$ of primary variables, while the fluid normal displacement U_n and the tractions t_k of the solid skeleton are gathered in the vector $\mathbf{t}$ of secondary variables. The fundamental solutions matrices can be divided into four submatrices: *00*, *0k*, *l0*, and *lk*; where the first index indicate where the load is applied, and the second index indicate where the response is being observed (*0*: fluid phase, $l,k=1,2$: solid skeleton). If $\mathbf{x}_i \in \Gamma$, then the integration domain is partitioned as $\Gamma=\lim_{\varepsilon\to 0^+}[(\Gamma-e^i)+\Gamma^i]$, being e^i the exclusion zone of Γ, and Γ^i an arc of radius ε that surrounds $\mathbf{x}_i$. Once the integration over Γ^i is done, the SBIE (1) turns into:

$$\mathbf{C}^i\mathbf{u}^i + \lim_{\varepsilon\to 0^+}\int_{\Gamma-e^i} \mathbf{T}^*\mathbf{u}\,\mathrm{d}\Gamma = \lim_{\varepsilon\to 0^+}\int_{\Gamma-e^i} \mathbf{U}^*\mathbf{t}\,\mathrm{d}\Gamma, \quad \mathbf{C}^i = \begin{bmatrix} Jc^i_{00} & 0 \\ 0 & c^i_{lk} \end{bmatrix} \tag{2}$$

where the free-terms c^i_{00} and c^i_{lk} are similar to those of the potential and the elastostatic problems, respectively. The components of the $\mathbf{U}^*$ fundamental solution matrix can be written as:

$$\tau^*_{00} = \frac{1}{2\pi}\eta, \quad \eta = \frac{1}{k_1^2-k_2^2}\left[\alpha_1 \mathrm{K}_0(ik_1r) - \alpha_2 \mathrm{K}_0(ik_2r)\right], \quad \alpha_j = k_j^2 - \frac{\mu}{\lambda+2\mu}k_3^2 \tag{3}$$

$$u^*_{0k} = -\frac{1}{2\pi}\Theta r_{,k}, \quad \Theta = \Theta_C\left[ik_1 \mathrm{K}_1(ik_1r) - ik_2 \mathrm{K}_1(ik_2r)\right], \quad \Theta_C = \left(\frac{Q}{R}-Z\right)\frac{1}{\lambda+2\mu}\frac{1}{k_1^2-k_2^2} \tag{4}$$

$$\tau^*_{l0} = \frac{1}{2\pi J}\Theta r_{,l} \tag{5}$$

$$\begin{gathered} u^*_{lk} = \frac{1}{2\pi\mu}\left(\psi\delta_{lk} - \chi r_{,l}r_{,k}\right), \quad \psi = \mathrm{K}_0(ik_3r) + \frac{1}{ik_3r}\mathrm{K}_1(ik_3r) - \frac{1}{k_1^2-k_2^2}\left[\frac{\beta_1}{ik_1r}\mathrm{K}_1(ik_1r) - \frac{\beta_2}{ik_2r}\mathrm{K}_1(ik_2r)\right] \\ \chi = \mathrm{K}_2(ik_3r) - \frac{1}{k_1^2-k_2^2}\left[\beta_1 \mathrm{K}_2(ik_1r) - \beta_2 \mathrm{K}_2(ik_2r)\right], \quad \beta_j = \frac{\mu}{\lambda+2\mu}k_j^2 - \frac{k_1^2k_2^2}{k_3^2} \end{gathered} \tag{6}$$

where $\mathrm{K}_n(z)$ is the modified Bessel function of the second kind, order n, and argument z. The components of the $\mathbf{T}^*$ fundamental solution matrix are obtained from:

$$U^*_{n00} + JX'^*_j n_j = \left(-Zu^*_{0j} - J\tau^*_{00,j}\right)n_j \tag{7}$$

$$t^*_{0k} = \left[\lambda u^*_{0m,m}\delta_{kj} + \mu\left(u^*_{0k,j} + u^*_{0j,k}\right) + \frac{Q}{R}\tau^*_{00}\delta_{kj}\right]n_j \tag{8}$$

$$U^*_{nl0} = \left(-Zu^*_{lj} - J\tau^*_{l0,j}\right)n_j \tag{9}$$

$$t_{lk}^{*}=\left[\lambda u_{lm,m}^{*}\delta_{kj}+\mu\left(u_{lk,j}^{*}+u_{lj,k}^{*}\right)+\frac{Q}{R}\tau_{l0}^{*}\delta_{kj}\right]n_{j} \tag{10}$$

Since τ_{00}^{*} and u_{lk}^{*} are written in a similar fashion to the scalar wave propagation and elastodynamic problems, respectively, it is easy to identify the *00* and *lk* submatrices of $\mathbf{U}^{*}$ and $\mathbf{T}^{*}$ with those of these problems. All integrals are regular or weakly singular, except those associated with t_{lk}^{*}, which are strongly singular and must be regularized by interpreting them in the Cauchy Principal Value sense. Their solution is well known, see for example [6].

The HBIE of the poroelastic problem is obtained by differentiating the SBIE (1) with respect to the coordinates of the collocation point, and then applying:

$$U_{n}^{i}=\left(-Zu_{j}^{i}-J\tau_{,j}^{i}\right)n_{j}^{i} \tag{11}$$

$$t_{l}^{i}=\left[\lambda u_{m,m}^{i}\delta_{lj}+\mu\left(u_{l,j}^{i}+u_{j,l}^{i}\right)+\frac{Q}{R}\tau^{i}\delta_{lj}\right]n_{j}^{i} \tag{12}$$

where $\mathbf{n}^{i}=\left(n_{1}^{i},n_{2}^{i}\right)$ is the unit normal at the collocation point. Then, the HBIE can be written as:

$$\begin{bmatrix}1 & 0\\ 0 & \delta_{lk}\end{bmatrix}\begin{Bmatrix}U_{n}^{i}\\ t_{k}^{i}\end{Bmatrix}+\int_{\Gamma}\begin{bmatrix}-s_{00}^{*} & s_{0k}^{*}\\ -s_{l0}^{*} & s_{lk}^{*}\end{bmatrix}\begin{Bmatrix}\tau\\ u_{k}\end{Bmatrix}\mathrm{d}\Gamma=\int_{\Gamma}\begin{bmatrix}-d_{00}^{*} & d_{0k}^{*}\\ -d_{l0}^{*} & d_{lk}^{*}\end{bmatrix}\begin{Bmatrix}U_{n}\\ t_{k}\end{Bmatrix}\mathrm{d}\Gamma,\quad \mathbf{C}^{i}\mathbf{t}^{i}+\int_{\Gamma}\mathbf{S}^{*}\mathbf{u}\,\mathrm{d}\Gamma=\int_{\Gamma}\mathbf{D}^{*}\mathbf{t}\,\mathrm{d}\Gamma \tag{13}$$

If $\mathbf{x}_{i}\in\Gamma$, then the integration domain is partitioned in a similar way as the SBIE, i.e. $\Gamma=\lim_{\varepsilon\to 0^{+}}[(\Gamma-e^{i})+\Gamma^{i}]$. However, some of the integrals associated with the matrix $\mathbf{S}^{*}$ are hypersingular integrals, which need certain continuity conditions in order to be solved. Let $I=\int_{A}^{B}F(x)/(x-x^{i})^{2}\,\mathrm{d}x,\ A<x<B$ be a hypersingular integral, if F belongs to the Hölder function space $C^{1,\alpha}$, the I exists in the Hadamard Finite Part sense. In order to fulfill this condition, it is necessary to impose that $\tau\left(\mathbf{x}^{i}\right)\in C^{1}$ and $u_{k}\left(\mathbf{x}^{i}\right)\in C^{1}$. Once the integration over Γ^{i} is done, the HBIE (13) turns into:

$$\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & \delta_{lk}\end{bmatrix}\begin{Bmatrix}U_{n}^{i}\\ t_{k}^{i}\end{Bmatrix}-\frac{1}{\pi}\left(\lim_{\varepsilon\to 0^{+}}\frac{1}{\varepsilon}\right)\begin{bmatrix}-J & 0\\ 0 & \dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\end{bmatrix}\begin{Bmatrix}\tau^{i}\\ u_{k}^{i}\end{Bmatrix}+\lim_{\varepsilon\to 0^{+}}\int_{\Gamma-e^{i}}\mathbf{S}^{*}\mathbf{u}\,\mathrm{d}\Gamma=\lim_{\varepsilon\to 0^{+}}\int_{\Gamma-e^{i}}\mathbf{D}^{*}\mathbf{t}\,\mathrm{d}\Gamma \tag{14}$$

where it has been assumed that $\Gamma(\mathbf{x}^{i})\in C^{1}$, i.e. the boundary is smooth at the collocation point. The integration over Γ^{i} has produced an unbounded term. However, it is cancelled by another unbounded term that emerges when the regularization process is performed to the integrals over $\Gamma-e^{i}$ associated with $\mathbf{S}^{*}$, resulting the Hadamard Finite Part of the original integral. Since the *00* and *lk* submatrices from $\mathbf{S}^{*}$ have the same kind of singularity of the fundamental solutions of the scalar wave propagation and elastodynamic problems, respectively, their regularization process is similar to those of these problems. The regularization process for the scalar wave propagation problem can be found in [1], and for the elastodynamic problem in [6,7]. The *0k* and *l0* submatrices of $\mathbf{S}^{*}$ contain strongly singular integrals similar to those that appear in the off-diagonal terms of the $\mathbf{T}^{*}$ matrix of the elastodynamic problem. The $\mathbf{D}^{*}$ fundamental solution matrix is similar to the $\mathbf{T}^{*}$ matrix, except for some signs and that $\mathbf{n}^{i}$ appears instead of $\mathbf{n}$.

SBIE/HBIE dual boundary formulation for poroelastic media

The SBIE/HBIE dual boundary formulation consists in the simultaneous collocation of the SBIE and the HBIE on the boundaries of null thickness inclusions, i.e. cracks, voids, or in our case, thin elastic bodies. It is the most direct and general approach to face problems with null thickness inclusions. Other techniques could be found in the introduction section of [4].

Let Γ be the boundary of a region Ω, resulting from the approaching of two identical boundaries, Γ^+ and Γ^-, whose normal vectors are pointing at each other, until they are coincident. The face Γ^+ is the reference face, thus $\mathbf{n}$ and $\mathbf{n}^i$ are defined on it. The variables of each face are indicated by $\square^+$ or $\square^-$. When the collocation point $\mathbf{x}_i \in \Gamma$, the integration domain for both the SBIE and the HBIE is (see Fig. 1):

$$\Gamma = \lim_{\varepsilon \to 0^+} \left\{ \left[\Gamma^+ - e^{i+} \right] + \Gamma^{i+} + \left[\Gamma^- - e^{i-} \right] + \Gamma^{i-} \right\} \tag{15}$$

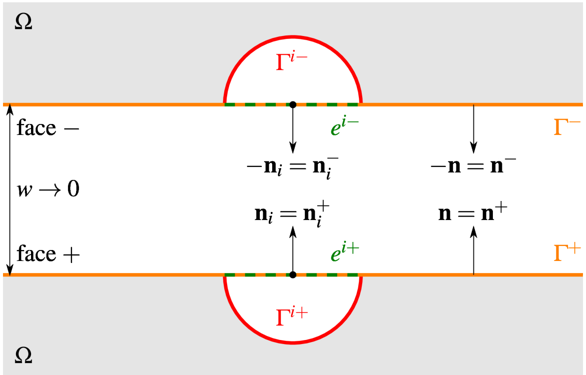

Figure 1: Integration domain for the dual boundary formulation

If it is assumed that $\Gamma(\mathbf{x}^i) \in C^1$, then the regularized SBIE/HBIE dual boundary formulation for poroelastic media can be written as:

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix} J & 0 \\ 0 & \delta_{lk} \end{bmatrix} \left(\mathbf{u}^{i+} + \mathbf{u}^{i-} \right) + \mathrm{CPV}\int_\Gamma \mathbf{T}^* \mathbf{u}\, \mathrm{d}\Gamma = \mathrm{RPV}\int_\Gamma \mathbf{U}^* \mathbf{t}\, \mathrm{d}\Gamma \\ &\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & \delta_{lk} \end{bmatrix} \left(\mathbf{t}^{i+} - \mathbf{t}^{i-} \right) + \mathrm{HPV}\int_\Gamma \mathbf{S}^* \mathbf{u}\, \mathrm{d}\Gamma = \mathrm{CPV}\int_\Gamma \mathbf{D}^* \mathbf{t}\, \mathrm{d}\Gamma \end{aligned} \tag{16}$$

where RPV (Riemann Principal Value), CPV (Cauchy Principal Value) and HFP (Hadamard Finite Part) before the integral operator sign is used to indicate that only the finite part of those integrals are considered. The integration and regularization process of those integrals is similar to the previously explained.

BEM-FEM coupling

The BEM dual boundary formulation together with a FEM model for the thin structures and appropriate coupling conditions make possible to build a BEM-FEM model for thin structures immersed or buried in inviscid fluid, viscoelastic and/or poroelastic media. The FEM model of the thin structures is not explained in this paper, but it could be found in [8,1]. In this paper, the coupling conditions when the surroundings is a poroelastic region are presented. The coupling conditions for inviscid fluids [1] and viscoelastic solids can be obtained as particular cases of the poroelastic case.

Let Υ_j be a BEM-FEM poroelastic soil – structure element composed by three sub-elements: Υ_j^+ and Υ_j^- (boundary elements), and Υ_j^s (finite element); see Fig. 2. The boundary elements are 3-noded quadratic line elements, where each node is associated with four variables: fluid phase normal displacement U_n, fluid phase equivalent stress τ, solid skeleton displacement vector $\mathbf{u}$, and solid skeleton traction vector $\mathbf{t}$. The finite element is a 3-noded straight beam element, and has eight degrees of freedom. The vertex nodes $i=1,2$ have translation $\mathbf{u}^{(i)}$ and rotation $\theta^{(i)}$, while the central node only has translation $\mathbf{u}^{(3)}$. The local axes are defined by the pair of vectors $\mathbf{x}'_1$ and $\mathbf{x}'_2$. Each node is associated with axial $s'^{(i)}_1$ and lateral $s'^{(i)}_2$ forces due to axial and lateral load distributions, respectively.

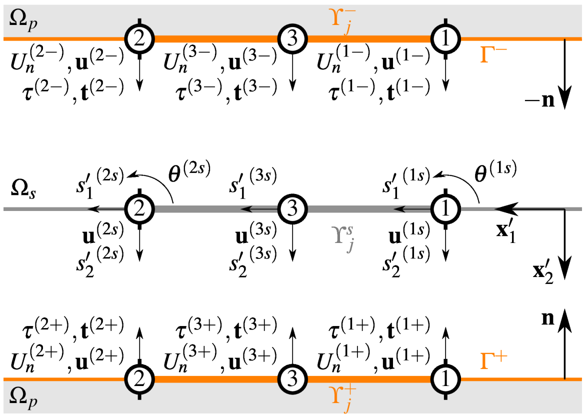

Figure 2: Coupling between sub-elements Υ_j^+, Υ_j^- and Υ_j^s (local numbering)

The compatibility equations for a node i are:

$$U_n^{(i+)} = \mathbf{u}^{(i+)} \cdot \mathbf{n}, \quad U_n^{(i-)} = -\mathbf{u}^{(i-)} \cdot \mathbf{n} \tag{17}$$

$$\mathbf{u}^{(i+)} = \mathbf{u}^{(is)}, \quad \mathbf{u}^{(i-)} = \mathbf{u}^{(is)} \tag{18}$$

where (17) establishes that both faces of the structure are impervious, and (18) establishes a perfect bonding between the soil and the structure. The equilibrium equation for a node i is:

$$\tau^{(i+)}\mathbf{n} + \mathbf{t}^{(i+)} - \tau^{(i-)}\mathbf{n} + \mathbf{t}^{(i-)} + s'^{(is)}_1\,\mathbf{x}'_1 + s'^{(is)}_2\,\mathbf{x}'_2 = \mathbf{0} \tag{19}$$

which is in reality two equations, one for each coordinate.

By examining Fig. 2, each vertex node has 17 unknowns in total, while the central node has 16 unknowns. The number of equations are: 3 (SBIE), 3 (HBIE), 3 for vertex nodes or 2 for the central node (FEM), 2 (impervious condition), 4 (perfect bonding), and 2 (equilibrium); in total 17 equations for vertex nodes, and 16 for the central node.

Validation of the BEM-FEM model

A modified problem from [9] is used to validate the model. The problem is a simplified water reservoir where bottom sediments support a wall (see Fig. 3). The upper part of the wall is immersed in the water and its base is buried in the bottom sediments. The reservoir has free tractions on the left and top boundaries, and horizontal displacements on the right and bottom boundaries. Four regions are present: dam wall Ω_1 (viscoelastic solid), bottom sediments Ω_2 (poroelastic solid), water Ω_3 (inviscid fluid), and the immersed/buried wall Ω_4 (viscoelastic solid). The viscoelastic regions Ω_1 and Ω_4 have the following properties: density $\rho = 2481.5\ \mathrm{kg/m^3}$, shear modulus $\mu = 11500\ \mathrm{MPa}$, Poisson's ratio $\nu = 0.20$ and damping coefficient $\xi = 0.05$. The poroelastic region Ω_2 has the following properties: fluid phase density $\rho_f = 1000\ \mathrm{kg/m^3}$, solid skeleton density $\rho_s = 2640\ \mathrm{kg/m^3}$, Lamé's first constant $\lambda = 17.9753\ \mathrm{MPa}$, $\mu = 7.7037\ \mathrm{MPa}$, damping coefficient $\xi = 0.05$, porosity $\phi = 0.60$, null added density, Biot's constants

$R = 1.24416 \cdot 10^9$ N/m^2 and $Q = 829.44 \cdot 10^6$ N/m^2, and dissipation constant $b = 3.5316 \cdot 10^6$ Ns/m^4. The fluid region Ω_3 has a density $\rho = 1000$ kg/m^3 and a wave propagation speed $c = 1438$ m/s.

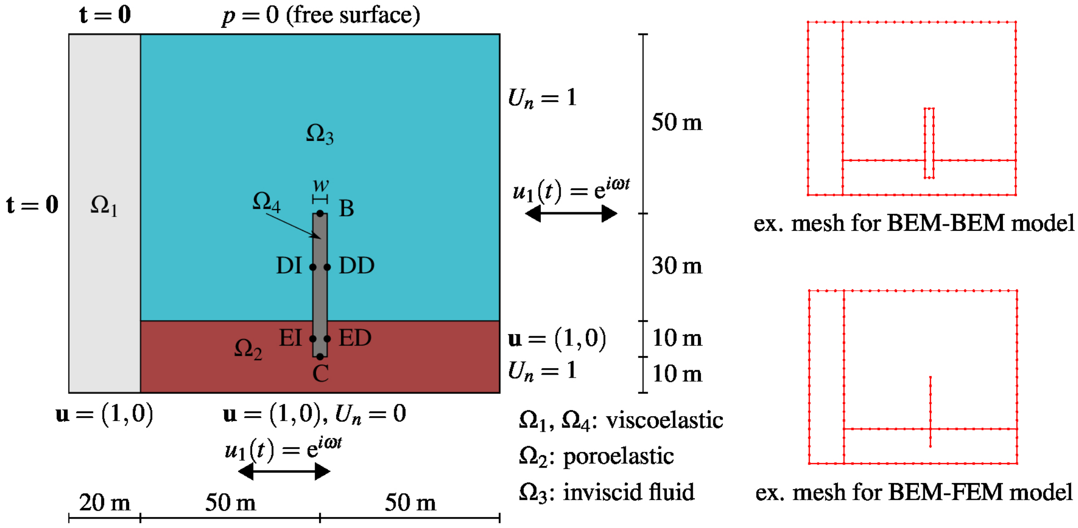

Figure 3: Water reservoir with bottom sediments supporting a wall

Three cases with three different widths of the immersed/buried wall are considered $w = \{1, 2, 5\}$ m, or in terms of the slenderness $L/w = \{40, 20, 8\}$. These cases are solved using the complete geometry using a BEM-BEM model, and using the developed BEM-FEM model for the immersed/buried wall. Note that the BEM-BEM model needs a different mesh for each case, while the BEM-FEM model only needs one mesh. The points B, C, DI, DD, EI and ED are selected points where several results are going to be plotted. In the plots, the normalized frequency ω/ω_1 is used, where $\omega_1 = 6.769$ rad/s is the first natural frequency of the dam wall on rigid foundation.

Fig. 4 and Fig. 5 show the u_1 amplification factor $\mathrm{abs}[(u_1 - 1)/1]$ at the point B and C, respectively. They show that the BEM-FEM model gets close to the BEM-BEM model as the slenderness of the wall increase. Even so, when the wall has slenderness $L/w = 8$ m, which is in the limit to consider it as a thin structure, the BEM-FEM model obtains a good reproduction of the response. Fig. 6 shows the pressure at the water at points DI and DD. Fig. 7 shows the fluid equivalent pressure at the bottom sediments at points EI and ED. Again, they show that the BEM-FEM model get close to the BEM-BEM model as wall thickness decrease.

Conclusions

A two-dimensional BEM-FEM model of thin structures for time harmonic analysis when they are surrounded by inviscid fluid, viscoelastic and/or poroelastic media has been presented. The key of the model is using the BEM dual boundary formulation for the surroundings, and a beam finite element for the thin structures. The model has been validated through a simple water reservoir problem, showing excellent agreement.

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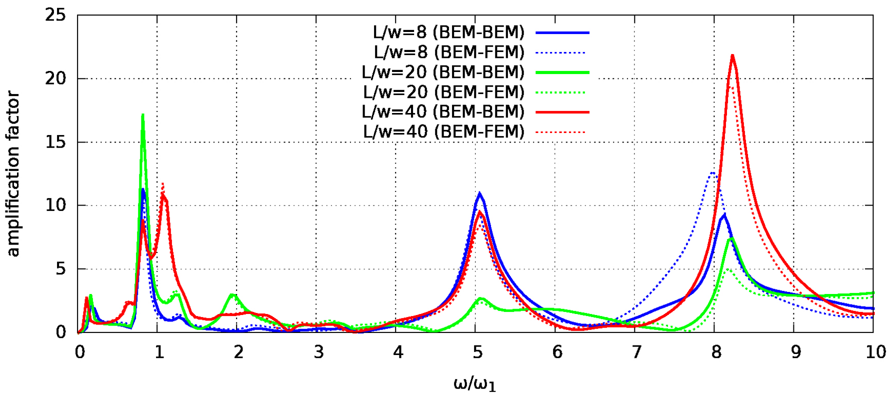

Figure 4: u_1 amplification factor at point B

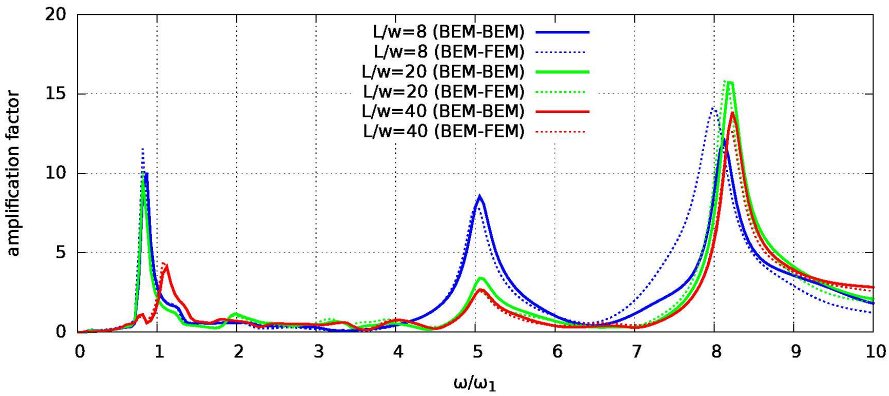

Figure 5: u_1 amplification factor at point C

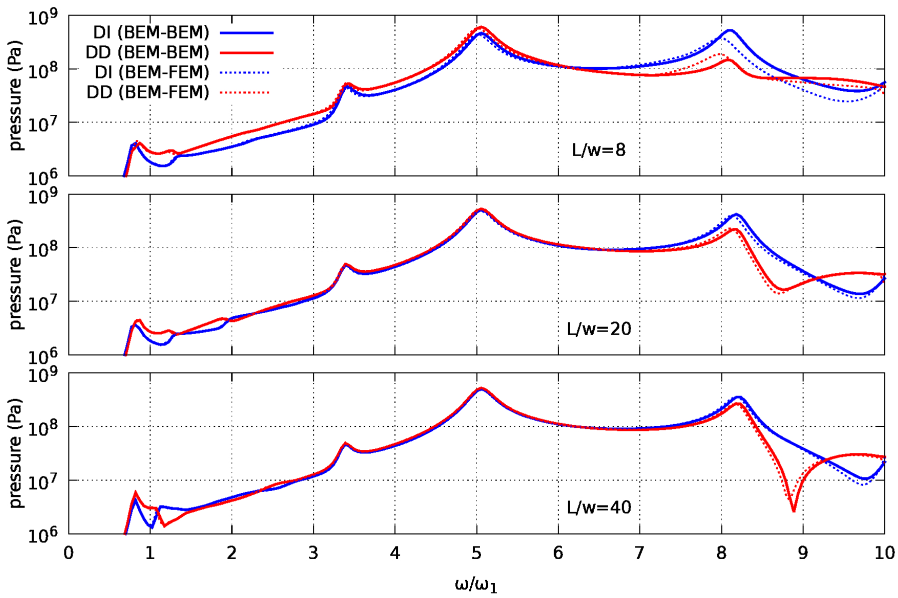

Figure 6: pressure p at points DI and DD

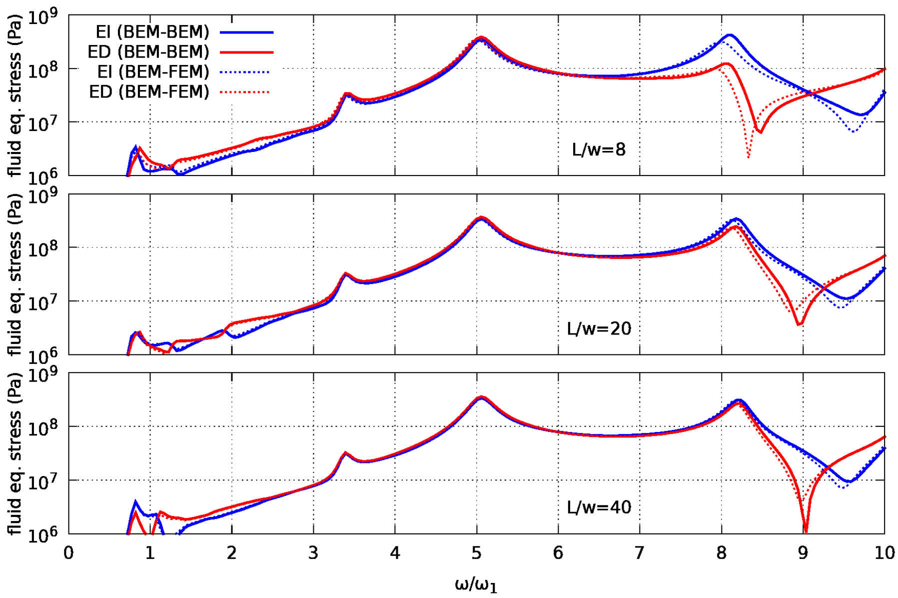

Figure 7: fluid equivalent stress τ at points EI and ED

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